

# PROOF OF THE KURLBERG-RUDNICK RATE CONJECTURE

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**Abstract.** In this paper we present a proof of the *Hecke quantum unique ergodicity conjecture* for the Berry-Hannay model, a model of quantum mechanics on a two dimensional torus. This conjecture was stated in Z. Rudnick's lectures at MSRI, Berkeley, 1999 and ECM, Barcelona, 2000.

**Résumé.** Nous proposons une démonstration de la conjecture d'unique ergodicité quantique d'Hecke pour le modèle de Berry-Hannay, un modèle de mécanique quantique sur un tore de dimension deux. Cette conjecture a été proposée par Z. Rudnick à MSRI, Berkeley, 1999 à l'ECM, Barcelona, 2000.

## 0 Introduction

**Hannay-Berry model.** In 1980 the physicists J. Hannay and Sir M.V. Berry [1] explore a model for quantum mechanics on the two dimensional symplectic torus  $(\mathbf{T}, \omega)$ .

**Quantum chaos.** Consider the ergodic discrete dynamical system on the torus, which is generated by an hyperbolic automorphism  $A \in \mathrm{SL}_2(\mathbb{Z})$ . Quantizing the system, we replace: the classical phase space  $(\mathbf{T}, \omega)$  by a Hilbert space  $\mathcal{H}_h$ , classical observables, i.e., functions  $f \in C^\infty(\mathbf{T})$ , by operators  $\pi_h(f) \in \mathrm{End}(\mathcal{H}_h)$  and classical symmetries by a unitary representation  $\rho_h : \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{U}(\mathcal{H}_h)$ . A fundamental meta-question in the area of quantum chaos is to *describe* the spectral properties of the quantum system  $\rho_h(A)$ , at least in the semi-classical limit as  $h \rightarrow 0$ .

**The rate conjecture.** In [5] Kurlberg and Rudnick proved that eigenvectors that satisfy certain additional symmetries of  $\rho_h(A)$  are semi-classically equidistributed with respect to the Haar measure on  $\mathbf{T}$ . In this paper we prove (see Theorem 3) the Kurlberg-Rudnick conjecture [6, 7] on the rate of convergence of the relevant distribution to the Haar measure.

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## 1 Classical Torus

Let  $(\mathbf{T}, \omega)$  be the two dimensional symplectic torus. Together with its linear symplectomorphisms  $\Gamma \simeq \mathrm{SL}_2(\mathbb{Z})$  it serves as a simple model of classical mechanics (a compact version of the phase space of the harmonic oscillator). More precisely, let  $\mathbf{T} = W/\Lambda$  where  $W$  is a two dimensional real vector space and  $\Lambda$  is a rank two unimodular lattice in  $W$ . We denote by  $\Lambda^* \subseteq W^*$  the dual lattice, i.e.,  $\Lambda^* = \{\xi \in W^* \mid \xi(\Lambda) \subset \mathbb{Z}\}$ . The lattice  $\Lambda^*$  is identified with the lattice of characters of  $\mathbf{T}$  by the map  $\xi \in \Lambda^* \longmapsto e^{2\pi i \langle \xi, \cdot \rangle} \in \mathbf{T}^\vee$ , where  $\mathbf{T}^\vee := \mathrm{Hom}(\mathbf{T}, \mathbb{C}^*)$ .

**Classical mechanical system.** We consider a very simple discrete mechanical system. An hyperbolic element  $A \in \Gamma$ , i.e.,  $|\mathrm{Tr}(A)| > 2$ , generates an ergodic discrete dynamical system on  $\mathbf{T}$ .

## 2 Quantization of the Torus

**The Weyl quantization model.** The Weyl quantization model works as follows. Let  $\mathcal{A}_\hbar$  be a one parameter deformation of the algebra  $\mathcal{A}$  of trigonometric polynomials on the torus. This algebra is known in the literature as the Rieffel torus [8]. The algebra  $\mathcal{A}_\hbar$  is constructed by taking the free algebra over  $\mathbb{C}$  generated by the symbols  $\{s(\xi) \mid \xi \in \Lambda^*\}$  and quotient out by the relation  $s(\xi + \eta) = e^{\pi i \hbar \omega(\xi, \eta)} s(\xi) s(\eta)$ . Here  $\omega$  is the form on  $W^*$  induced by the original form  $\omega$  on  $W$ . The algebra  $\mathcal{A}_\hbar$  contains as a standard basis the lattice  $\Lambda^*$ . Therefore, one can identify the algebras  $\mathcal{A}_\hbar \simeq \mathcal{A}$  as vector spaces. Hence, every function  $f \in \mathcal{A}$  can be viewed as an element of  $\mathcal{A}_\hbar$ . For a fixed  $\hbar$  a representation  $\pi_\hbar : \mathcal{A}_\hbar \longrightarrow \text{End}(\mathcal{H}_\hbar)$  serves as a quantization protocol.

**Equivariant Weyl quantization of the torus.** The group  $\Gamma$  acts on the lattice  $\Lambda^*$ , therefore it acts on  $\mathcal{A}_\hbar$ . For an element  $B \in \Gamma$ , we denote by  $f \longmapsto f^B$  the action of  $B$  on an element  $f \in \mathcal{A}_\hbar$ . Let  $\Gamma_p \simeq \text{SL}_2(\mathbb{F}_p)$  denotes the quotient group of  $\Gamma$  modulo  $p$ .

**Theorem 2.1 (Canonical equivariant quantization)** *Let  $\hbar = \frac{1}{p}$ , where  $p$  is an odd prime. There exists a unique (up to isomorphism) pair of representations  $\pi_\hbar : \mathcal{A}_\hbar \longrightarrow \text{End}(\mathcal{H}_\hbar)$  and  $\rho_\hbar : \Gamma \longrightarrow \text{GL}(\mathcal{H}_\hbar)$  satisfying the compatibility condition (Egorov identity)  $\rho_\hbar(B) \pi_\hbar(f) \rho_\hbar(B)^{-1} = \pi_\hbar(f^B)$ , where  $\pi_\hbar$  is an irreducible representation and  $\rho_\hbar$  is a representation of  $\Gamma$  that factors through the quotient group  $\Gamma_p$ .*

**Quantum mechanical system.** Let  $(\pi_\hbar, \rho_\hbar, \mathcal{H}_\hbar)$  be the canonical equivariant quantization. Let  $A$  be our fixed hyperbolic element, considered as an element of  $\Gamma_p$ . The element  $A$  generates a quantum dynamical system. For every (pure) quantum state  $v \in S(\mathcal{H}_\hbar) = \{v \in \mathcal{H}_\hbar : \|v\| = 1\}$ ,  $v \longmapsto v^A := \rho_\hbar(A)v$ .

## 3 Hecke Quantum Unique Ergodicity

Denote by  $T_A$  the centralizer of  $A$  in  $\Gamma_p \simeq \text{SL}_2(\mathbb{F}_p)$ . We call  $T_A$  the *Hecke torus* (cf. [5]). The precise statement of the **Kurlberg-Rudnick conjecture** (cf. [4] and [6, 7]) is given in the following theorem:

**Theorem 3.1 (Hecke Quantum Unique Ergodicity)** *Let  $\hbar = \frac{1}{p}$ ,  $p$  an odd prime. For every  $f \in \mathcal{A}_\hbar$  and  $v \in S(\mathcal{H}_\hbar)$ , we have:*

$$\left| \mathbf{A} \mathbf{v}_{T_A} (\langle v | \pi_\hbar(f) v \rangle) - \int_{T_A} f \omega \right| \leq \frac{C_f}{\sqrt{p}}, \quad (3.0.1)$$

where  $\mathbf{A} \mathbf{v}_{T_A} (\langle v | \pi_\hbar(f) v \rangle) := \sum_{B \in T_A} \langle v | \pi_\hbar(f^B) v \rangle$  is the average with respect to the group  $T_A$  and  $C_f$  is an explicit constant depending only on  $f$ .

## 4 Proof of the Hecke Quantum Unique Ergodicity Conjecture

It is enough to prove the conjecture for the case when  $f$  is a non-trivial character  $\xi \in \Lambda^*$  and  $v$  is an Hecke eigenvector with eigencharacter  $\chi : T_A \longrightarrow \mathbb{C}^*$ . In this case Theorem 3.1 can be restated in the form:

**Theorem 4.1 (Hecke Quantum Unique Ergodicity (Restated))** *Let  $\hbar = \frac{1}{p}$ , where  $p$  is an odd prime. For every  $\xi \in \Lambda^*$  and every character  $\chi : T_A \longrightarrow \mathbb{C}^*$  the following holds:*

$$\left| \sum_{B \in T_A} \text{Tr}(\rho_\hbar(B) \pi_\hbar(\xi)) \chi(B) \right| \leq 2\sqrt{p}.$$

**The trace function.** Denote by  $F$  the function  $F : \Gamma \times \Lambda^* \longrightarrow \mathbb{C}$  defined by  $F(B, \xi) = \text{Tr}(\rho(B) \pi_\hbar(\xi))$ . We denote by  $V := \Lambda^*/p\Lambda^*$  the quotient vector space, i.e.,  $V \simeq \mathbb{F}_p^2$ . The symplectic form  $\omega$  specializes to give a symplectic form on  $V$ . The group  $\Gamma_p$  is the group of linear symplectomorphisms of  $V$ , i.e.,  $\Gamma_p = \text{Sp}(V, \omega)$ . Set  $Y_0 := \Gamma \times \Lambda^*$  and  $Y := \Gamma_p \times V$ . We have a natural quotient map  $Y_0 \longrightarrow Y$ .

**Lemma 4.2** *The function  $F : Y_0 \longrightarrow \mathbb{C}$  factors through the quotient  $Y$ .*

From now on  $Y$  will be considered as the default domain of the function  $F$ . The function  $F : Y \longrightarrow \mathbb{C}$  is invariant with respect to the action of  $\Gamma_p$  on  $Y$  given by the following formula:

$$\begin{aligned} \Gamma_p \times Y &\xrightarrow{\alpha} Y, \\ (S, (B, \xi)) &\longrightarrow (SBS^{-1}, S\xi). \end{aligned} \tag{4.0.2}$$

**Geometrization (Sheaffication).** Next, we will phrase a geometric statement that will imply Theorem 4.1. Moving into the geometric setting, we replace the set  $Y$  by an algebraic variety and the functions  $F$  and  $\chi$  by sheaf theoretic objects, also of a geometric flavor.

**Step 1.** The set  $Y$  is the set of rational points of an algebraic variety  $\mathbb{Y}$  defined over  $\mathbb{F}_p$ . To be more precise,  $\mathbb{Y} \simeq \mathbb{S}p \times \mathbb{V}$ . The variety  $\mathbb{Y}$  is equipped with an endomorphism  $\text{Fr} : \mathbb{Y} \longrightarrow \mathbb{Y}$  called Frobenius. The set  $Y$  is identified with the set of fixed points of Frobenius  $Y = \mathbb{Y}^{\text{Fr}} = \{y \in \mathbb{Y} : \text{Fr}(y) = y\}$ . Finally, we denote by  $\alpha$  the algebraic action of  $\mathbb{S}p$  on the variety  $\mathbb{Y}$  (cf. (4.0.2)).

**Step 2.** The following theorem proposes an appropriate sheaf theoretic object standing in place of the function  $F : Y \longrightarrow \mathbb{C}$ . Denote by  $\mathcal{D}_{c,w}^b(\mathbb{Y})$  the bounded derived category of constructible  $\ell$ -adic Weil sheaves on  $\mathbb{Y}$ .

**Theorem 4.3 (Geometrization Theorem)** *There exists an object  $\mathcal{F} \in \mathcal{D}_{c,w}^b(\mathbb{Y})$  satisfying the following properties:*

1. (Function) It is associated, via the *sheaf-to-function correspondence*, to the function  $F : Y \longrightarrow \mathbb{C}$ , i.e.,  $f^{\mathcal{F}} = F$ .
2. (Weight) It is of weight  $w(\mathcal{F}) \leq 0$ .
3. (Equivariance) For every element  $S \in \mathbb{S}p$  there exists an isomorphism  $\alpha_S^* \mathcal{F} \simeq \mathcal{F}$ .
4. (Formula) On introducing coordinates  $\mathbb{V} \simeq \mathbb{A}^2$  we identify  $\mathbb{S}p \simeq \mathbb{S}L_2$ . Then there exists an isomorphism  $\mathcal{F}|_{\mathbb{T} \times \mathbb{V}} \simeq \mathcal{L}_{\psi(\frac{1}{2}\lambda\mu\frac{a+1}{a-1})} \otimes \mathcal{L}_{\sigma(a)}$ .<sup>1</sup>  
Here  $\mathbb{T} := \{(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix})\}$  stands for the standard torus,  $(\lambda, \mu)$  are the coordinates on  $\mathbb{V}$  and  $\mathcal{L}_{\psi}$ ,  $\mathcal{L}_{\sigma}$  the Artin-Schreier and Kummer sheaves.

**Geometric statement.** Fix an element  $\xi \in \Lambda^*$  with  $\xi \neq 0$ . We denote by  $i_{\xi}$  the inclusion map  $i_{\xi} : T_A \times \xi \longrightarrow Y$ . Going back to Theorem 4.1 and putting its content in a functorial notation, we write the following inequality:

$$\left| pr_!(i_{\xi}^*(F) \cdot \chi) \right| \leq 2\sqrt{p}.$$

In words, taking the function  $F : Y \longrightarrow \mathbb{C}$  and restricting  $F$  to  $T_A \times \xi$  and get  $i_{\xi}^*(F)$ . Multiply  $i_{\xi}^*F$  by the character  $\chi$  to get  $i_{\xi}^*(F) \cdot \chi$ . Integrate  $i_{\xi}^*(F) \cdot \chi$  to the point, this means to sum up all its values, and get a scalar  $a_{\chi} := pr_!(i_{\xi}^*(F) \cdot \chi)$ . Here  $pr$  stands for the projection  $pr : T_A \times \xi \longrightarrow pt$ . Then Theorem 4.1 asserts that the scalar  $a_{\chi}$  is of an absolute value less than  $2\sqrt{p}$ .

Repeat the same steps in the geometric setting. We denote again by  $i_{\xi}$  the closed imbedding  $i_{\xi} : T_A \times \xi \longrightarrow \mathbb{Y}$ . Take the sheaf  $\mathcal{F}$  on  $\mathbb{Y}$  and apply the following sequence of operations. Pull-back  $\mathcal{F}$  to the closed subvariety  $T_A \times \xi$  and get the sheaf  $i_{\xi}^*(\mathcal{F})$ . Take the tensor product of  $i_{\xi}^*(\mathcal{F})$  with the Kummer sheaf  $\mathcal{L}_{\chi}$  and get  $i_{\xi}^*(\mathcal{F}) \otimes \mathcal{L}_{\chi}$ . Integrate  $i_{\xi}^*(\mathcal{F}) \otimes \mathcal{L}_{\chi}$  to the point and get the sheaf  $pr_!(i_{\xi}^*(\mathcal{F}) \otimes \mathcal{L}_{\chi})$  on the point.

Recall  $w(\mathcal{F}) \leq 0$ . Knowing that the Kummer sheaf has weight  $w(\mathcal{L}_{\chi}) \leq 0$  we deduce that  $w(i_{\xi}^*(\mathcal{F}) \otimes \mathcal{L}_{\chi}) \leq 0$ .

**Theorem 4.4 (Deligne, Weil II [3])** *Let  $\pi : \mathbb{X}_1 \longrightarrow \mathbb{X}_2$  be a morphism of algebraic varieties. Let  $\mathcal{L} \in \mathcal{D}_{c,w}^b(\mathbb{X}_1)$  be a sheaf of weight  $w(\mathcal{L}) \leq w$  then  $w(\pi_!(\mathcal{L})) \leq w$ .*

<sup>1</sup>By this we mean that  $\mathcal{F}|_{\mathbb{T} \times \mathbb{V}}$  is isomorphic to the extension of the sheaf defined by the formula in the right-hand side.

Using Theorem 4.4 we get  $w(pr_!(i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi)) \leq 0$ .

Now, consider the sheaf  $\mathcal{G} := pr_!(i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi)$ . It is an object in  $\mathcal{D}_{c,w}^b(pt)$ . The sheaf  $\mathcal{G}$  is associated by Grothendieck's Sheaf-To-Function correspondence to the scalar  $a_\chi$ :

$$a_\chi = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(\text{Fr}|_{H^i(\mathcal{G})}). \quad (4.0.3)$$

Finally, we can give the geometric statement about  $\mathcal{G}$ , which will imply Theorem 4.1.

**Lemma 4.5 (Vanishing Lemma)** *Let  $\mathcal{G} = pr_!(i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi)$ . All cohomologies  $H^i(\mathcal{G})$  vanish except for  $i = 1$ . Moreover,  $H^1(\mathcal{G})$  is a two dimensional vector space.*

Theorem 4.1 now follows easily. By Lemma 4.5 only the first cohomology  $H^1(\mathcal{G})$  does not vanish and it is two dimensional. Having that  $w(\mathcal{G}) \leq 0$  implies that the eigenvalues of Frobenius acting on  $H^1(\mathcal{G})$  are of absolute value  $\leq \sqrt{p}$ . Hence, using formula (4.0.3) we get  $|a_\chi| \leq 2\sqrt{p}$ .

**Proof of the Vanishing Lemma. Step 1.** All tori in  $\text{Sp}$  are conjugated. On introducing coordinates, i.e.,  $\mathbb{V} \simeq \mathbb{A}^2$ , we make the identification  $\text{Sp} \simeq \text{SL}_2$ . In these terms there exists an element  $S \in \text{SL}_2$  conjugating the Hecke torus  $\mathbb{T}_A \subset \text{SL}_2$  with the standard torus  $\mathbb{T} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \subset \text{SL}_2$ , namely  $S\mathbb{T}_A S^{-1} = \mathbb{T}$ .

**Step 2.** Using the equivariance property of the sheaf  $\mathcal{F}$  (see Theorem 4.3, property 3) we see that it is sufficient to prove the Vanishing Lemma for the sheaf  $\mathcal{G}_{st} := pr_!(i_\eta^* \mathcal{F} \otimes \alpha_{S!} \mathcal{L}_\chi)$ , where  $\eta = S \cdot \xi$  and  $\alpha_S$  is the restriction of the action  $\alpha$  to the element  $S$ .

**Step 3.** The Vanishing Lemma holds for the sheaf  $\mathcal{G}_{st}$ . We write  $\eta = (\lambda, \mu)$ . By Theorem 4.3 Property 4 we have  $i_\eta^* \mathcal{F} \simeq \mathcal{L}_{\psi(\frac{1}{2} \lambda \mu \frac{a+1}{a-1})} \otimes \mathcal{L}_{\sigma(a)}$ , where  $a$  is the coordinate of the standard torus  $\mathbb{T}$  and  $\lambda \cdot \mu \neq 0^2$ . The sheaf  $\alpha_{S!} \mathcal{L}_\chi$  is a character sheaf on the torus  $\mathbb{T}$ . A direct computation proves the Vanishing Lemma. ■

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<sup>2</sup>This is a direct consequence of the fact that  $A \in \text{SL}_2(\mathbb{Z})$  is an hyperbolic element and does not have eigenvectors in  $\Lambda^*$ .